

One-loop Integral Coefficients from Generalized Unitarity

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I describe a method for determining the coefficients of scalar integrals for one-loop amplitudes in quantum field theory. The method is based upon generalized unitarity and the behavior of amplitudes when the free parameters of the cut momenta approach infinity. The method works for arbitrary masses of both external and internal legs of the amplitudes. It therefore applies not only to QCD but also to the Electroweak theory and to quantum field theory in general.

I. INTRODUCTION

For many years, high energy physics has been looking forward to the beginning of the LHC program. There are high expectations that experiment will answer a number of outstanding theoretical issues, such as the nature of electroweak symmetry breaking and the resolution of the hierarchy problem. However, the success of the experimental program will depend upon the development of better theoretical tools. Our ability to identify new physics at the TeV scale will depend, in large part, on our ability to accurately describe and separate out the known physics of the Standard Model.

It is well established that at the Tevatron and other colliders that leading order calculations of QCD processes are insufficient for an accurate description of hadronic interactions. Next-to-leading order (NLO) corrections in QCD are often quite large. The corrections typically change both the predicted magnitude of the scattering process and the shapes of distributions.

At the Tevatron, which has little reach into the multi-hundred GeV range, it has been found that computing NLO QCD corrections is generally sufficient to reduce the theoretical uncertainty of a calculation to about 10%. At the LHC, however, with its reach extending above the TeV scale, it has been found that electroweak corrections can become quite large. It is therefore becoming important to develop tools for performing higher-order corrections in the full $SU(3) \otimes SU(2) \otimes U(1)$ Standard Model.

In the past, many of the most exotic computational methods focused on QCD not just because QCD corrections are large even at relatively low energies, but also because working with massless quarks and gluons simplifies expressions sufficiently to allow complicated algorithms to be worked out. Recent advances now provide a framework in which massive theories can access the sophisticated advances developed for QCD calculations.

The workhorse in perturbative calculations has long been the Feynman diagram approach, which approach systematically includes all perturbative effects. However, the rapid growth in the number of Feynman diagrams with the number of external legs, even for tree-level amplitudes, limits the practical application of this technique. In addition, Feynman diagram calculations are subject to large cancellations among

individual diagrams, especially when working with gauge theories.

One tool that has long been used in QCD calculations to obtain compact results and largely avoid unphysical singularities and gauge cancellations is the helicity method [1, 2, 3, 4, 5, 6, 7, 8]. When combined with recursion relations, both off-shell [9, 10] and, more recently, on-shell [11, 12, 13, 14] the helicity method can be used to generate compact expressions for very complicated scattering processes.

Another very powerful technique that has greatly simplified the calculation of loop amplitudes in QCD is the unitarity method [15, 16, 17, 18, 19, 20, 21, 22]. An essential feature of the unitarity method is that it sews together tree-level amplitudes into loop amplitudes. Thus, efficient techniques for computing tree-level amplitudes which give compact expressions, like the use of recursion relations and the helicity method, directly benefit the unitarity method of computing loop amplitudes.

There have been a number of new developments that have significantly enhanced the power of recursive methods and the unitarity method. These include the use of maximally helicity-violating (MHV) vertices in recursion relations at tree-level [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and in loops [34, 35, 36, 37]. Unitarity methods have been improved by the use of the holomorphic anomaly [38, 39, 40] to evaluate cuts and the use of complex momenta [41, 42, 43, 44, 45] within the context of generalized unitarity [46, 47, 48, 49, 50, 51, 52], which allows for the use of multiple cuts in a single amplitude.

These techniques can be combined into a “unitarity bootstrap” [53, 54], a systematic, recursive approach to making high-multiplicity QCD calculations practical. The bootstrap combines the use of four-dimensional unitarity to determine the logarithmic and polylogarithmic terms in the loop amplitude with on-shell recursion relations to determine the rational contributions.

Recently, Forde [55] has described an efficient method for extracting the coefficients of the triangle and bubble functions in amplitudes where only massless particles circulate in the loops. Combined with the result of Ref. [41], this is sufficient to determine the cut-constructable part of one-loop QCD amplitudes. In this paper, I will extend Forde’s formalism and define methods to extract the coefficients of all loop-integral functions in massive as well as massless theo-

ries. This will then allow this method to be used in the full $SU(3) \otimes SU(2) \otimes U(1)$ Standard Model and even in the Supersymmetric Standard Model.

The plan of the paper is as follows: In Section (II), I will discuss the notation used and the application of spinor helicity conventions to massive complex momenta. In Section (III), I will present an overview of the formalism which allows the simple extraction of the loop integral functions. In Section (IV), I will discuss the extraction of the box functions; in Section (V) the extraction of the triangle functions; in Section (VI), the bubble functions; and in Section (VII), the tadpoles. Finally, I will comment on my results and draw my conclusions.

II. NOTATION AND CONVENTIONS

An important feature of the method described in this paper is that it involves on-shell amplitudes where some of the on-shell momenta are complex. Therefore, I will describe the conventions used to handle complex momenta and, especially, the spinorial representations of complex momenta.

A. Spinor Representations of Real Momenta

I treat real momenta using the spinor-helicity formalism of Mangano and Parke [8], which I will summarize briefly. Let p^μ be a massless momentum in Minkowski space, and $\psi(p)$ the Dirac spinor representing a massless fermion of momentum p ,

$$\not{p} \psi(p) = 0, \quad p^2 = 0. \quad (1)$$

The two helicity states of $\psi(p)$ are given by

$$\begin{aligned} \psi_\pm(p) &= \frac{1}{2} (1 \pm \gamma_5) \psi(p) \equiv |p^\pm\rangle, \\ \bar{\psi}_\pm(p) &= \frac{1}{2} \bar{\psi}(p) (1 \pm \gamma_5) \equiv \langle p^\pm|. \end{aligned} \quad (2)$$

The phase convention is chosen such that

$$|p^\pm\rangle = |p^\mp\rangle^c, \quad \langle p^\pm| = {}^c \langle p^\mp|, \quad (3)$$

where the c indicates charge conjugation. To render expressions more compact, I adopt the notation

$$\begin{aligned} \langle pq \rangle &= \langle p^- | q^+ \rangle, \\ [pq] &= \langle p^+ | q^- \rangle. \end{aligned} \quad (4)$$

Some important identities are:

$$\begin{aligned} \langle p^+ | q^+ \rangle &= \langle p^- | q^- \rangle = 0, \\ \langle pp \rangle &= [pp] = 0, \\ \langle pq \rangle &= -\langle qp \rangle \quad [pq] = -[qp], \\ \langle pq \rangle [qp] &= 2 p \cdot q, \\ \langle p^\pm | \gamma^\mu | p^\pm \rangle &= 2 p^\mu, \\ \not{p} &= |p^+\rangle \langle p^+| + |p^-\rangle \langle p^-|, \\ \gamma_\mu \langle p^\pm | \gamma^\mu | q^\pm \rangle &= 2 (|p^\mp\rangle \langle q^\mp| + |q^\pm\rangle \langle p^\pm|), \\ \langle AB \rangle \langle CD \rangle &= \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle CB \rangle, \\ [AB] [CD] &= [AC] [BD] + [AD] [CB]. \end{aligned} \quad (5)$$

The last two identities are known as the Schouten identities.

This formalism can alternatively be expressed in terms of Weyl-van der Waerden (WvdW) spinors [56]. Indeed there is a one-to-one correspondence between the spinors defined here and WvdW spinors:

$$\begin{aligned} |\phi^+\rangle &\leftrightarrow \begin{pmatrix} \phi_A \\ 0 \end{pmatrix}, & \langle \phi^-| &\leftrightarrow (\phi^A \quad 0), \\ |\chi^-\rangle &\leftrightarrow \begin{pmatrix} 0 \\ \chi^{\dot{A}} \end{pmatrix}, & \langle \chi^+| &\leftrightarrow (0 \quad \chi_{\dot{A}}), \\ \langle \phi \chi \rangle_{MP} &\leftrightarrow \langle \phi \chi \rangle_{WvdW}, & [\chi \phi]_{MP} &\leftrightarrow \langle \phi \chi \rangle_{WvdW}^*, \\ \gamma^\mu &\leftrightarrow \begin{pmatrix} 0 & \sigma_{AB}^\mu \\ \sigma_{\mu, \dot{A} \dot{B}} & 0 \end{pmatrix}, \end{aligned} \quad (6)$$

Where the MP and $WvdW$ subscripts denote the ‘‘Mangano & Parke’’ and ‘‘Weyl-van der Waerden’’ notation for spinor products, respectively.

1. Fermion Wave Functions for Real Massless Momenta

These are essentially trivial, since the notation is defined in terms of massless Dirac spinors. The only point of convention is that, for the sake of defining helicity, all external particles are taken to be outgoing. This means that we identify

$$|p^\pm\rangle = v_\pm(p), \quad \langle p^\pm| = \bar{u}_\pm(p). \quad (7)$$

The phase convention defined in Eq. (3) means that

$$u_\pm(p) = v_\mp(p), \quad \bar{u}_\pm(p) = \bar{v}_\mp(p). \quad (8)$$

2. Massless Vector Boson Wave-functions for Real Momenta

Massless spin-1 particles have two physical polarization states. The standard practice in spinor helicity methods is to use the light-like axial gauge, in which the polarization vectors are defined in terms of both the momentum vector k and a reference vector g . The gauge invariance of the spin-1 field manifests itself in the arbitrariness of the reference momentum. For an outgoing vector field of momentum k , the helicity states are can be written as

$$\varepsilon^{+\mu}(k; g) = \frac{\langle k^+ | \gamma^\mu | g^+ \rangle}{\sqrt{2} \langle gk \rangle}, \quad \varepsilon^{-\mu}(k; g) = \frac{\langle k^- | \gamma^\mu | g^- \rangle}{\sqrt{2} [kg]}. \quad (9)$$

These polarization vectors have the usual properties

$$\begin{aligned}
(\varepsilon^\pm)^* &= \varepsilon^\mp, \\
\varepsilon^\pm \cdot \varepsilon^\pm &= 0, \\
\varepsilon^\pm \cdot \varepsilon^\mp &= -1, \\
\varepsilon^{+\mu} \varepsilon^{-\nu} + \varepsilon^{-\mu} \varepsilon^{+\nu} &= -g^{\mu\nu} + \frac{k^\mu g^\nu + g^\mu k^\nu}{g \cdot k}
\end{aligned} \tag{10}$$

The arbitrariness of the choice of g can be seen by examining the difference between two choices of g . Consider contracting a polarization vector with some random vector,

$$\begin{aligned}
&(\varepsilon^{+\mu}(k; g_1) - \varepsilon^{+\mu}(k; g_2)) \langle a^- | \gamma_\mu | b^- \rangle \\
&= \sqrt{2} \left(\frac{[kb] \langle ag_1 \rangle}{\langle g_1 k \rangle} - \frac{[kb] \langle ag_2 \rangle}{\langle g_2 k \rangle} \right) \\
&= \sqrt{2} [kb] \frac{\langle ag_1 \rangle \langle g_2 k \rangle - \langle ag_2 \rangle \langle g_1 k \rangle}{\langle g_1 k \rangle \langle g_2 k \rangle} \\
&= \sqrt{2} [kb] \frac{\langle ak \rangle \langle g_2 g_1 \rangle}{\langle g_1 k \rangle \langle g_2 k \rangle} \\
&= \frac{k^\mu \langle g_1 g_2 \rangle}{\sqrt{2} \langle g_1 k \rangle \langle g_2 k \rangle} \langle a^- | \gamma_\mu | b^- \rangle.
\end{aligned} \tag{11}$$

Thus, the difference between the polarization vectors generated by two choices of g is proportional to k^μ . Since gauge bosons couple to conserved currents, this is a pure gauge term and does not contribute to the amplitude.

B. Massive Real Momenta

A massive real momentum can be represented as the sum of two massless momenta. There is great freedom in choosing this decomposition, which leads to a variety of choices for representing massive fermion and vector wave functions. The construction of helicity basis wave functions for massive particles is discussed thoroughly in Ref. [56] and is not repeated here. One can even choose the decomposition so that the massive wave functions are eigenstates of the helicity projection operator.

C. Massless Complex Momenta

The massless complex momenta that I will encounter will be defined in terms of the spinor representations of two real massless momenta χ^μ and ψ^μ ,

$$\ell^\mu = y \chi^\mu + w \psi^\mu + \frac{t}{2} \langle \chi^- | \gamma^\mu | \psi^- \rangle + \frac{wy}{2t} \langle \psi^- | \gamma^\mu | \chi^- \rangle. \tag{12}$$

One can define a spinor representation of this complex momentum as

$$\begin{aligned}
\langle \ell^+ | &= \frac{y}{t} \langle \chi^+ | + \langle \psi^+ |, & \langle \ell^- | &= t \langle \chi^- | + w \langle \psi^- |, \\
| \ell^- \rangle &= \frac{y}{t} | \chi^- \rangle + | \psi^- \rangle, & | \ell^+ \rangle &= t | \chi^+ \rangle + w | \psi^+ \rangle.
\end{aligned} \tag{13}$$

Note that the different helicity states are not related by complex conjugation, as they are in the case of real momenta. As in the case of real momenta, however, these spinor representations can be used directly as massless fermion wave function and in massless vector wave-functions (see Section (II A)).

D. Massive Complex Momenta

Let us assume that we have an on-shell complex momentum ℓ^μ with mass m parametrized in terms of real massless four-momenta χ^μ and ψ^μ ,

$$\begin{aligned}
\ell^\mu &= y \chi^\mu + w \psi^\mu + \frac{t}{2} \langle \chi^- | \gamma^\mu | \psi^- \rangle \\
&\quad + \left(\frac{wy}{2t} - \frac{m^2}{4t \chi \cdot \psi} \right) \langle \psi^- | \gamma^\mu | \chi^- \rangle.
\end{aligned} \tag{14}$$

I can trivially decompose ℓ^μ into two massless complex momenta,

$$\begin{aligned}
\ell^\mu &= \ell_1^\mu + \ell_2^\mu, \\
\ell_1^\mu &= y \chi^\mu + w \psi^\mu + \frac{t}{2} \langle \chi^- | \gamma^\mu | \psi^- \rangle + \frac{wy}{2t} \langle \psi^- | \gamma^\mu | \chi^- \rangle, \\
\ell_2^\mu &= -\frac{m^2}{4t \chi \cdot \psi} \langle \psi^- | \gamma^\mu | \chi^- \rangle,
\end{aligned} \tag{15}$$

and then find spinor representations of ℓ_1^μ and ℓ_2^μ ,

$$\begin{aligned}
\langle \ell_1^+ | &= \frac{y}{t} \langle \chi^+ | + \langle \psi^+ | & \langle \ell_1^- | &= t \langle \chi^- | + w \langle \psi^- | \\
| \ell_1^- \rangle &= \frac{y}{t} | \chi^- \rangle + | \psi^- \rangle & | \ell_1^+ \rangle &= t | \chi^+ \rangle + w | \psi^+ \rangle \\
\langle \ell_2^+ | &= \frac{m}{[\chi \psi]} \langle \chi^+ | & \langle \ell_2^- | &= \frac{m}{t \langle \chi \psi \rangle} \langle \psi^- | \\
| \ell_2^- \rangle &= \frac{m}{[\chi \psi]} | \chi^- \rangle & | \ell_2^+ \rangle &= \frac{m}{t \langle \chi \psi \rangle} | \psi^+ \rangle, \\
\ell &= | \ell_1^+ \rangle \langle \ell_1^+ | + | \ell_1^- \rangle \langle \ell_1^- | + | \ell_2^+ \rangle \langle \ell_2^+ | + | \ell_2^- \rangle \langle \ell_2^- |.
\end{aligned} \tag{16}$$

With this parametrization, one immediately finds that

$$\langle \ell_1 \ell_2 \rangle = -\langle \ell_2 \ell_1 \rangle = m, \quad [\ell_2 \ell_1] = -[\ell_1 \ell_2] = m. \tag{17}$$

1. Fermion Wave Functions

If the internal massive particle is a fermion, we must construct wave functions which obey the Dirac equation. Equation (17) which implies that

$$\begin{aligned}
\ell | \ell_1^+ \rangle &= -m | \ell_2^- \rangle & \ell | \ell_2^- \rangle &= -m | \ell_1^+ \rangle, \\
\ell | \ell_1^- \rangle &= m | \ell_2^+ \rangle & \ell | \ell_2^+ \rangle &= m | \ell_1^- \rangle, \\
\langle \ell_1^+ | \ell &= -m \langle \ell_2^- | & \langle \ell_2^- | \ell &= -m \langle \ell_1^+ |, \\
\langle \ell_1^- | \ell &= m \langle \ell_2^+ | & \langle \ell_2^+ | \ell &= m \langle \ell_1^- |,
\end{aligned} \tag{18}$$

which means that the Dirac spinors can be written as:

$$\begin{aligned} |u_\uparrow(\ell)\rangle &= |\ell_1^+\rangle - |\ell_2^-\rangle & |u_\downarrow(\ell)\rangle &= |\ell_2^+\rangle + |\ell_1^-\rangle, \\ |v_\uparrow(\ell)\rangle &= -|\ell_2^+\rangle + |\ell_1^-\rangle & |v_\downarrow(\ell)\rangle &= |\ell_1^+\rangle + |\ell_2^-\rangle, \\ \langle \bar{u}_\uparrow(\ell)| &= \langle \ell_1^+| + \langle \ell_2^-| & \langle \bar{u}_\downarrow(\ell)| &= \langle \ell_2^+| + \langle \ell_1^-|, \\ \langle \bar{v}_\uparrow(\ell)| &= -\langle \ell_2^+| + \langle \ell_1^-| & \langle \bar{v}_\downarrow(\ell)| &= \langle \ell_1^+| + \langle \ell_2^-|. \end{aligned} \quad (19)$$

Note that I label the spin states \uparrow / \downarrow , rather than \pm . This indicates that the decomposition of ℓ into ℓ_1 and ℓ_2 defined above does not yield wave functions that are eigenstates of the helicity projection operator. Since states with complex momenta are necessarily internal states, I do not need helicity projections, I only need to sum over the spin states. One can easily verify that these spinors obey the Dirac Equation,

$$\begin{aligned} \ell |u_\uparrow(\ell)\rangle &= m |u_\uparrow(\ell)\rangle & \ell |u_\downarrow(\ell)\rangle &= m |u_\downarrow(\ell)\rangle, \\ \ell |v_\uparrow(\ell)\rangle &= -m |v_\uparrow(\ell)\rangle & \ell |v_\downarrow(\ell)\rangle &= -m |v_\downarrow(\ell)\rangle, \\ \langle \bar{u}_\uparrow(\ell)| \ell &= m \langle \bar{u}_\uparrow(\ell)| & \langle \bar{u}_\downarrow(\ell)| \ell &= m \langle \bar{u}_\downarrow(\ell)|, \\ \langle \bar{v}_\uparrow(\ell)| \ell &= -m \langle \bar{v}_\uparrow(\ell)| & \langle \bar{v}_\downarrow(\ell)| \ell &= -m \langle \bar{v}_\downarrow(\ell)|, \end{aligned} \quad (20)$$

the standard normalization conditions,

$$\begin{aligned} \langle \bar{u}_i(\ell) | u_j(\ell) \rangle &= 2m \delta_{ij} & \langle \bar{u}_i(\ell) | v_j(\ell) \rangle &= 0 \quad i, j \in \{\uparrow \downarrow\} \\ \langle \bar{v}_i(\ell) | v_j(\ell) \rangle &= -2m \delta_{ij} & \langle \bar{v}_i(\ell) | u_j(\ell) \rangle &= 0 \end{aligned} \quad (21)$$

and combine to form the standard projection operator,

$$\begin{aligned} \sum_{i \in \{\uparrow \downarrow\}} |u_i(\ell)\rangle \langle \bar{u}_i(\ell)| &= \sum_{j \in \{+-\}} \left(|\ell_1^j\rangle \langle \ell_1^j| + |\ell_2^j\rangle \langle \ell_2^j| \right) \\ &+ |\ell_2^+\rangle \langle \ell_1^-| + |\ell_1^-\rangle \langle \ell_2^+| - |\ell_1^+\rangle \langle \ell_2^-| - |\ell_2^-\rangle \langle \ell_1^+| \\ &= \ell + m, \\ \sum_{i \in \{\uparrow \downarrow\}} |v_i(\ell)\rangle \langle \bar{v}_i(\ell)| &= \sum_{j \in \{+-\}} \left(|\ell_1^j\rangle \langle \ell_1^j| + |\ell_2^j\rangle \langle \ell_2^j| \right) \\ &- |\ell_2^+\rangle \langle \ell_1^-| - |\ell_1^-\rangle \langle \ell_2^+| + |\ell_1^+\rangle \langle \ell_2^-| + |\ell_2^-\rangle \langle \ell_1^+| \\ &= \ell - m, \end{aligned} \quad (22)$$

where I can use the Schouten identities to show that

$$|\ell_2^+\rangle \langle \ell_1^-| + |\ell_1^-\rangle \langle \ell_2^+| - |\ell_1^+\rangle \langle \ell_2^-| - |\ell_2^-\rangle \langle \ell_1^+| = m. \quad (23)$$

$$\begin{aligned} -\langle a\ell_1 | \ell_2 b \rangle + \langle a\ell_2 | \ell_1 b \rangle &= -\langle ab | \ell_2 \ell_1 \rangle = m \langle ab \rangle \\ -[a\ell_2][b\ell_1] + [a\ell_1][\ell_2 b] &= -[ab][\ell_1 \ell_2] = m [ab]. \end{aligned} \quad (24)$$

In this way, I have defined orthogonal spin states for massive fermions with complex momenta. Although they are defined in terms of momentum spinors of definite helicity, these spin states are not eigenstates of the helicity projector.

2. Massive Gauge Boson Wave Functions

Massive gauge bosons have three physical spin states. For outgoing particles, the wave functions, or polarization vectors, can be written as

$$\begin{aligned} \varepsilon^{\uparrow\mu} &= \frac{\langle \ell_1^+ | \gamma^\mu | \ell_2^+ \rangle}{\sqrt{2} \langle \ell_2 \ell_1 \rangle} = -\frac{\langle \ell_1^+ | \gamma^\mu | \ell_2^+ \rangle}{\sqrt{2} m} \\ \varepsilon^{\downarrow\mu} &= \frac{\langle \ell_1^- | \gamma^\mu | \ell_2^- \rangle}{\sqrt{2} [\ell_1 \ell_2]} = -\frac{\langle \ell_1^- | \gamma^\mu | \ell_2^- \rangle}{\sqrt{2} m} \\ \varepsilon^{\bullet\mu} &= \frac{1}{m} (\ell_1^\mu - \ell_2^\mu) \end{aligned} \quad (25)$$

As with the fermionic wave-functions, I do not use the standard symbols of $\{+ - 0\}$ since these spin states are not eigenvectors of the helicity projector. They are, however, orthonormal and display all of the usual properties expected of polarization vectors:

$$\begin{aligned} \varepsilon^\uparrow \cdot \varepsilon^\uparrow &= 0 & \varepsilon^\uparrow \cdot \varepsilon^\downarrow &= -1 & \varepsilon^\uparrow \cdot \varepsilon^\bullet &= 0, \\ \varepsilon^\downarrow \cdot \varepsilon^\uparrow &= -1 & \varepsilon^\downarrow \cdot \varepsilon^\downarrow &= 0 & \varepsilon^\downarrow \cdot \varepsilon^\bullet &= 0, \\ \varepsilon^\bullet \cdot \varepsilon^\uparrow &= 0 & \varepsilon^\bullet \cdot \varepsilon^\downarrow &= 0 & \varepsilon^\bullet \cdot \varepsilon^\bullet &= -1, \end{aligned} \quad (26)$$

$$\varepsilon^{\uparrow\mu} \varepsilon^{\downarrow\nu} + \varepsilon^{\downarrow\mu} \varepsilon^{\uparrow\nu} + \varepsilon^{\bullet\mu} \varepsilon^{\bullet\nu} = -g^{\mu\nu} + \frac{\ell^\mu \ell^\nu}{m^2}$$

E. Scalar Loop Integrals

To establish my sign convention, I will define the scalar loop integrals as follows,

$$\begin{aligned} F_0 &= (\ell^2 - m_0^2), \\ F_1 &= ((\ell + K_1)^2 - m_1^2), \\ F_2 &= ((\ell + K_2)^2 - m_2^2), \\ F_3 &= ((\ell + K_3)^2 - m_3^2), \\ D_0(m_0; K_1, m_1; K_2, m_2; K_3, m_3) &= -i(4\pi)^{D/2} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{F_0 F_1 F_2 F_3}, \\ C_0(m_0; K_1, m_1; K_2, m_2) &= -i(4\pi)^{D/2} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{F_0 F_1 F_2}, \\ B_0(m_0; K_1, m_1) &= -i(4\pi)^{D/2} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{F_0 F_1}, \\ A_0(m_0) &= -i(4\pi)^{D/2} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{F_0}. \end{aligned} \quad (27)$$

Note the sign convention on the momenta in the propagators. If, because of routing or some other convention, one of the momenta in the denominator is negative, say $F_1 = ((\ell - K_1)^2 - m_1^2)$, the loop integral functions would be written as $B_0(m_0, -K_1, m_1)$, $C_0(m_0, -K_1, m_1, K_2, m_2)$, etc.

III. METHODS

In theories of with only massless particles propagating in loops, like QCD, it has been shown that any one-loop integral can be decomposed into a sum of scalar box, triangle and bubble loop integral functions that can be constructed from cuts using unitarity and a set of rational terms [16]

$$A_{n,\text{QCD}}^{1\text{-loop}} = \mathcal{R}_n + \frac{\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}} \left(\sum_i b_i B_0^{(i)} + \sum_j c_j C_0^{(j)} + \sum_k d_k D_0^{(k)} \right), \quad (28)$$

where $B_0^{(i)}$ represent the bubble integrals, $C_0^{(j)}$ the triangle integrals and $D_0^{(k)}$ the boxes. The fact that only box functions are needed is because higher point functions can be written as sums of the boxes formed by pinching vertices together [57, 58]. The rational terms, \mathcal{R}_n cannot obtained from cuts, but can derived from on-shell recursion relations [11, 12, 13, 14]

In theories with massive particles, this formula must be augmented to include tadpole functions, $A_0^{(m)}$

$$A_n^{1\text{-loop}} = \mathcal{R}_n + \frac{\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}} \left(\sum_m a_m A_0^{(m)} + \sum_i b_i B_0^{(i)} + \sum_j c_j C_0^{(j)} + \sum_k d_k D_0^{(k)} \right). \quad (29)$$

In standard Passarino-Veltman reduction [59], one works in $D = 4 - 2\epsilon$ dimensions and the coefficients of the loop integral functions depend on the dimensional regulator ϵ . Rational terms develop when ϵ -dependent pieces of the coefficients multiply poles in ϵ from the loop integral function. In the methods described below, the cuts will be evaluated in $D = 4$ dimensions, thereby missing those rational terms, which are assigned to \mathcal{R}_n . \mathcal{R}_n will then be determined by loop-level on-shell recursion relations as part of the unitarity bootstrap.

The basic idea of the unitarity method is that when one cuts a one-loop amplitude, the terms on opposite sides of the cuts are tree-level amplitudes. These tree level amplitudes can be efficiently computed using helicity methods. However, helicity methods become very messy when they are extended into $D = 4 - 2\epsilon$ dimensions. It is desirable, therefore, to perform cuts in $D = 4$ dimensions. Doing so, however, misses various rational terms, as described above, which must be accounted for somehow. In pure QCD, a detailed knowledge of the soft and collinear factorization properties of amplitudes is sufficient to construct the missing rational terms. In massive theories, however, it is possible that there are rational terms that cannot be constructed in this way [17]. The development of on-shell recursion techniques [11, 12, 13, 14] to compute the rational terms allows the reliable use of four dimensional unitarity cuts in massive theories as well.

In generalized unitarity, multiple cuts are made on the loop amplitude, dividing it into multiple tree amplitudes. The method I will describe below uses generalized unitarity to

determine the coefficients of loop integrals in terms of the tree-level amplitudes that appear at the vertices of the cut diagrams. Different approaches to determining loop integral coefficients include algebraic solution [60, 61]), using D -dimensional cuts [62, 63], and using standard unitarity cuts via the holomorphic anomaly [64].

IV. FOUR PARTICLE CUTS AND THE EXTRACTION OF SCALAR BOX COEFFICIENTS

The use of quadruple cuts, shown in figure 1, to extract the box coefficient was first derived in Ref. [41]. I re-derive the

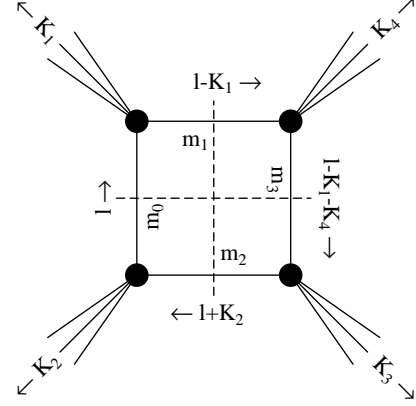


FIG. 1: a quadruple cut.

result here, using a more efficient momentum parametrization and extending the procedure to explicitly permit masses on the internal legs. The basic procedure is to cut all four internal legs, and put the cut legs on shell,

$$\frac{i}{(\ell + K_i)^2 - m_i^2} \longrightarrow (2\pi) \delta((\ell + K_i)^2 - m_i^2) \quad (30)$$

Imposing the cut conditions,

$$\begin{aligned} \ell^2 - m_0^2 &= 0, & (\ell - K_1)^2 - m_1^2 &= 0, \\ (\ell + K_2)^2 - m_2^2 &= 0, & (\ell - K_1 - K_4)^2 - m_3^2 &= 0, \end{aligned} \quad (31)$$

singles out a unique box configuration. Other boxes satisfy a different set of cuts and lower point diagrams cannot satisfy all four cuts. The contribution of this box topology to the one-loop amplitude is obtained by sewing together the four tree-level diagrams formed by cutting the loop propagators (See figure 2.)

In $D = 4$ dimensions, the four delta-function constraints completely determine the loop momentum ℓ^μ . There are two solutions and, in general, ℓ is complex valued. The coefficient d_m , corresponding to scalar box $D_0^{(m)}$, is given by averaging the product of the four tree-level diagrams, evaluated with the solutions to the loop momentum,

$$d_k = \frac{i}{2} \sum_{i=1}^2 A_1^{(k)}(\ell_i) A_2^{(k)}(\ell_i) A_3^{(k)}(\ell_i) A_4^{(k)}(\ell_i). \quad (32)$$

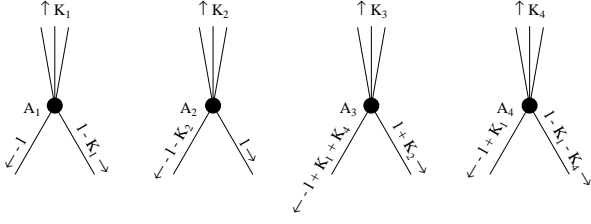


FIG. 2: Tree level amplitudes contributing to a particular box topology.

A. Parametrizing the Cut Loop Momentum

It is convenient to parametrize ℓ in terms of the spinor representations of two massless real momenta which can be constructed from any two adjacent external momenta on the box. (This parametrization was previously used in Refs. [60, 61, 65].) I first choose two adjacent external momenta, K_1 and K_2 and project them onto one another to form two massless momenta, K_1^\flat and K_2^\flat .

$$\begin{aligned} K_1 &= K_1^\flat + \frac{S_1}{\gamma_{12}} K_2^\flat, & K_2 &= K_2^\flat + \frac{S_2}{\gamma_{12}} K_1^\flat, \\ K_1^\flat &= \frac{K_1 - \frac{S_1}{\gamma_{12}} K_2}{1 - \frac{S_1 S_2}{\gamma_{12}^2}}, & K_2^\flat &= \frac{K_2 - \frac{S_2}{\gamma_{12}} K_1}{1 - \frac{S_1 S_2}{\gamma_{12}^2}}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} S_1 &= K_1 \cdot K_1, & S_2 &= K_2 \cdot K_2, \\ \gamma_{12} &= 2 K_1^\flat \cdot K_2^\flat = K_1 \cdot K_2 \pm \sqrt{\Delta(K_1, K_2)}, \\ \Delta(K_1, K_2) &= (K_1 \cdot K_2)^2 - S_1 S_2. \end{aligned} \quad (34)$$

Note that there are two solutions for γ_{12} unless either $S_1 = 0$ or $S_2 = 0$.

Assuming that the Gram determinant, $\Delta(K_1, K_2)$, does not vanish, I can now form four massless vectors to use as a basis for solving for the loop momentum ℓ .

$$\begin{aligned} a_1^\mu &= K_1^\flat{}^\mu, & a_2^\mu &= K_2^\flat{}^\mu, \\ a_3^\mu &= \langle K_1^\flat - |\gamma^\mu| K_2^\flat \rangle, & a_4^\mu &= \langle K_2^\flat - |\gamma^\mu| K_1^\flat \rangle, \\ \ell^\mu &= \alpha_1 a_1^\mu + \alpha_2 a_2^\mu + \alpha_3 a_3^\mu + \alpha_4 a_4^\mu. \end{aligned} \quad (35)$$

This basis has the property that $a_3 \cdot a_4 = 4 a_1 \cdot a_2 = 2 \gamma_{12}$, but all other inner products of the a_i among themselves vanish. The

solutions for ℓ are:

$$\begin{aligned} \alpha_1 &= \frac{S_2 (\gamma_{12} - S_1) + (\gamma_{12} - S_2) m_0^2 - \gamma_{12} m_2^2 + S_2 m_1^2}{\gamma_{12}^2 - S_1 S_2}, \\ \alpha_2 &= \frac{S_1 (\gamma_{12} - S_2) + (\gamma_{12} - S_1) m_0^2 - \gamma_{12} m_1^2 + S_1 m_2^2}{\gamma_{12}^2 - S_1 S_2}, \\ \beta_3 &= 2 (\alpha_1 - 1) K_1^\flat \cdot K_4 + 2 \left(\alpha_2 - \frac{S_1}{\gamma_{12}} \right) K_2^\flat \cdot K_4 \\ &\quad - S_4 + m_3^2 - m_1^2, \\ \beta_4 &= \alpha_1 \alpha_2 - \frac{m_0^2}{\gamma_{12}}, \\ \alpha_3 &= - \frac{\beta_3 \pm \sqrt{\beta_3^2 - 2 \beta_4 \text{Tr} K_1^\flat K_4 K_2^\flat K_4}}{2 \langle K_1^\flat - |K_4| K_2^\flat \rangle}, \\ \alpha_4 &= \frac{\beta_4}{4 \alpha_3}. \end{aligned} \quad (36)$$

There are two solutions for α_3 and it might appear that, combined with the two solutions for γ_{12} , there are four solutions for ℓ . However, it works out that

$$\begin{aligned} \ell^\mu(\gamma_{12}^+, \alpha_3^+) &= \ell^\mu(\gamma_{12}^-, \alpha_3^-), \\ \ell^\mu(\gamma_{12}^+, \alpha_3^-) &= \ell^\mu(\gamma_{12}^-, \alpha_3^+), \end{aligned} \quad (37)$$

so that there are only two solutions for ℓ^μ .

This parametrization of ℓ^μ looks rather different than that in Ref. [41], which used as a basis the three independent external momenta, K_1^μ , K_2^μ and K_4^μ , and the antisymmetric combination of those three, $P^\mu = \epsilon^{\mu\nu\rho\lambda} K_{1\nu} K_{2\rho} K_{4\lambda}$. Numerically, of course, the solutions are identical, but that given here has a number of features to recommend it. First, it is quite compact and easy to compute, even allowing for arbitrary values for the internal and external masses. Second, it is better behaved in Gram-singular configurations. As with all integral reduction techniques, we find spurious singularities in the form of inverse powers of the Gram determinant arising from tensor reduction. The Gram determinants, $\Delta(K_1, K_2, K_4)$, are found in the on-shell solution to the loop momentum ℓ . Numerical analysis shows that ℓ scales like $1/\sqrt{\Delta(K_1, K_2, K_4)}$ near the Gram singularity. This property is explicit in Eq. (36), where the denominator of α_3 , $\langle K_1^\flat - |K_4| K_2^\flat \rangle$ is the (complex) square root of $\Delta(K_1, K_2, K_4)$. In the solution of Ref. [41], the components of ℓ have coefficients that scale like $1/P^2$, where $P^2 = \Delta(K_1, K_2, K_4)$. Since ℓ actually scales like $1/\sqrt{\Delta(K_1, K_2, K_4)}$, there are cancellations hidden in the parametrization. This makes the problem of locating and canceling these spurious singularities much more difficult.

V. THREE PARTICLE CUTS AND THE EXTRACTION OF SCALAR TRIANGLE COEFFICIENTS

The procedure for extracting the box coefficients is so remarkably simple that one is inspired to try the approach for

extracting the coefficients of lower point integrals. The use of triple cuts in one loop amplitudes to extract the triangle coefficient was first discussed by Mastrolia [66]. A complication arises when one imposes a triple cut on a one-loop amplitude. One does pick out a single triangle topology, but one also gets contributions from box topologies formed by splitting one of the triangle's vertices to open up a fourth propagator. Since the box contributions can be determined by the prescription above, one needs a way of separating the pure triangle contributions from the already known box terms. Forde [55] has recently described an elegant way of doing so. The discussion below follows that of Forde, but allows for massive internal legs on the triangles.

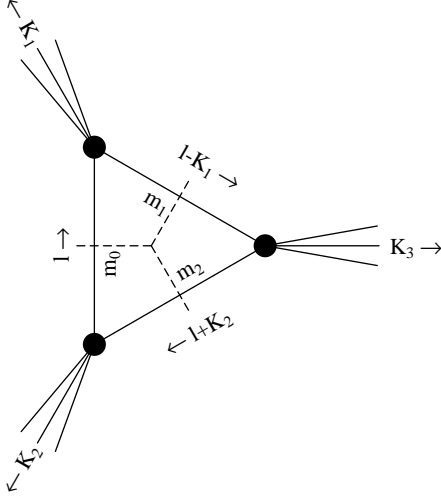


FIG. 3: a triple cut.

I consider a triple-cut triangle, as shown in figure 3. The three delta function constraints imposed by the cuts,

$$\delta(\ell^2 - m_0^2), \delta((\ell - K_1)^2 - m_1^2), \delta((\ell + K_2)^2 - m_2^2), \quad (38)$$

are not sufficient to completely fix the loop momentum, ℓ , which must therefore have an unconstrained degree of freedom. Using the same basis involving K_1^b and K_2^b as for the box, I can parametrize the loop momentum as

$$\begin{aligned} \ell^\mu &= \alpha_1 a_1^\mu + \alpha_2 a_2^\mu + \frac{t}{2} a_3^\mu + \frac{\alpha_4}{2t} a_4^\mu, \\ \alpha_1 &= \frac{S_2(\gamma_{12} - S_1) + (\gamma_{12} - S_2)m_0^2 - \gamma_{12}m_2^2 + S_2m_2^2}{\gamma_{12}^2 - S_1S_2}, \\ \alpha_2 &= \frac{S_1(\gamma_{12} - S_2) + (\gamma_{12} - S_1)m_0^2 - \gamma_{12}m_1^2 + S_1m_2^2}{\gamma_{12}^2 - S_1S_2}, \\ \alpha_4 &= \alpha_1\alpha_2 - \frac{m_0^2}{\gamma_{12}}. \end{aligned} \quad (39)$$

The integral depicted in figure 3 is given by

$$\begin{aligned} &c_j C_0(m_0, -K_1, m_1, K_2, m_2) \\ &= i \int \frac{d^4\ell}{(2\pi)^4} \frac{A_1^{(j)}(K_1; \ell) A_2^{(j)}(K_2; \ell) A_3^{(j)}(K_3; \ell)}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)((\ell + K_2)^2 - m_2^2)} \\ &\rightarrow i(-2\pi i)^3 \int \frac{d^4\ell}{(2\pi)^4} A_1^{(j)}(K_1; \ell) A_2^{(j)}(K_2; \ell) A_3^{(j)}(K_3; \ell) \times \\ &\quad \delta(\ell^2 - m_0^2) \delta((\ell - K_1)^2 - m_1^2) \delta((\ell + K_2)^2 - m_2^2) \\ &= i(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t A_1^{(j)}(t) A_2^{(j)}(t) A_3^{(j)}(t), \end{aligned} \quad (40)$$

where J_t is the Jacobian of the transformation from the delta-function constrained integral over $d^4\ell$ to the integral over the remaining free parameter t . Treating t as a complex variable and partial fractioning off terms with poles at finite t , this last integral can be rewritten as

$$\begin{aligned} c_j &= i(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t \left(\left[\text{Inf}_t A_1^{(j)} A_2^{(j)} A_3^{(j)} \right] (t) \right. \\ &\quad \left. + \sum_{\{k\}} \left[\frac{\text{Res}_{t=t_k} A_1^{(j)} A_2^{(j)} A_3^{(j)}}{t - t_k} \right] \right), \end{aligned} \quad (41)$$

which represents a sum over the residues of all poles $\{k\}$ at finite t_k and a contribution at infinity. The $[\text{Inf}_t]$ term is defined so that

$$\lim_{t \rightarrow \infty} ([\text{Inf}_t A_1 A_2 A_3](t) - A_1(t) A_2(t) A_3(t)) = 0. \quad (42)$$

The $[\text{Inf}_t]$ term will be some polynomial in t ,

$$[\text{Inf}_t A_1 A_2 A_3](t) = \sum_{n=0}^m e_n t^n, \quad (43)$$

where m is set by the maximum tensor rank allowed. For renormalizable theories, the maximum tensor rank for the triangle is three.

Now, recall from the discussion of the box coefficient that the fourth delta function constraint fixed the value of t at some complex value t_0 . Thus, the sum over residues at finite t simply correspond to the contributions to the triple-cut from the various box configurations that satisfy those three cuts. Moreover, the terms only give contributions to the scalar box coefficients. The triangle contributions of the triple-cut come exclusively from the terms at infinity. That is,

$$c_j = i(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t [\text{Inf}_t A_1^{(j)} A_2^{(j)} A_3^{(j)}](t). \quad (44)$$

So now, I need to construct a table for integrals of the form

$$i(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t t^n. \quad (45)$$

This is easily done by considering tensor triangle integrals of

the form

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot a_4)^n}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)((\ell + K_2)^2 - m_2^2)} \rightarrow (-2\pi i)^3 \gamma_{12}^n \int \frac{dt}{(2\pi)^4} J_t t^n \quad (46)$$

For the scalar integral, $n = 0$, it is clear that

$$i(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t = 1. \quad (47)$$

For tensor triangles however, with $n > 0$, the integrals vanish. By Lorentz invariance, the components of the tensor triangles must involve only products of K_1 , K_2 or the metric tensor $g^{\mu\nu}$. But K_1 and K_2 decompose into K_1^\flat and K_2^\flat , which annihilate a_4^μ via the Dirac equation, and metric tensor terms vanish because $a_4 \cdot a_4 = 0$. Therefore,

$$i(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t t^{n>0} = 0, \quad (48)$$

and

$$c_j = - \left[\text{Inf}_t A_1^{(j)} A_2^{(j)} A_3^{(j)} \right] (t) |_{t=0}. \quad (49)$$

VI. TWO PARTICLE CUTS AND THE EXTRACTION OF SCALAR BUBBLE COEFFICIENTS

To extract the coefficients of bubble integrals, I proceed as before and impose the cuts that define the bubble topology

$$\delta(\ell^2 - m_0^2), \quad \delta((\ell - K_1)^2 - m_1^2). \quad (50)$$

Only one bubble configuration will satisfy these cuts, but multiple triangle and box configurations will do so. Since the boxes and triangles can be extracted by the methods above, the task here is to isolate the pure bubble contributions.

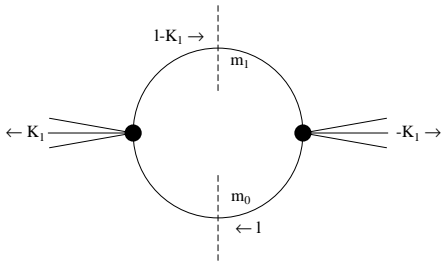


FIG. 4: a double cut.

Since I only have one external momentum, K_1 , in a bubble configuration, I will choose an arbitrary massless momentum χ^μ to define my parametrization. My massless projection of K_1 and the set of basis vectors for the loop momentum is then

defined in terms of χ^μ :

$$\begin{aligned} K_1 &= K_1^\flat + \frac{S_1}{\gamma_{1\chi}} \chi, & \gamma_{1\chi} &= 2 K_1 \cdot \chi = 2 K_1^\flat \cdot \chi, \\ a_1^\mu &= K_1^{\flat\mu}, & a_2^\mu &= \chi^\mu, \\ a_3^\mu &= \langle K_1^\flat - |\gamma^\mu| \chi^- \rangle, & a_4^\mu &= \langle \chi^- |\gamma^\mu| K_1^\flat \rangle, \\ \ell^\mu &= y a_1^\mu + \alpha_2 a_2^\mu + \frac{t}{2} a_3^\mu + \alpha_4 a_4^\mu, \end{aligned} \quad (51)$$

where both y and t are free parameters. The other coefficients, α_2 and α_4 are fixed by the cut conditions and are found to be

$$\alpha_2 = \frac{S_1 + m_0^2 - m_1^2}{\gamma_{1\chi}} - \frac{S_1}{\gamma_{1\chi}} y, \quad \alpha_4 = \frac{1}{2t} \left(\alpha_2 y - \frac{m_0^2}{\gamma_{1\chi}} \right). \quad (52)$$

The integral depicted in figure 4 is

$$\begin{aligned} & b_i B_0(m_0, -K_1, m_1) \\ &= i \int \frac{d^4 \ell}{(2\pi)^4} \frac{A_1^{(i)}(K_1; \ell) A_2^{(i)}(K_2; \ell)}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)} \\ &\rightarrow i(-2\pi i)^2 \int \frac{d^4 \ell}{(2\pi)^4} A_1^{(i)}(K_1; \ell) A_2^{(i)}(K_2; \ell) \times \\ &\quad \delta(\ell^2 - m_0^2) \delta((\ell - K_1)^2 - m_1^2) \\ &= i(-2\pi i)^2 \int \frac{dt dy}{(2\pi)^4} J_{t,y} A_1^{(i)}(t, y) A_2^{(i)}(t, y), \end{aligned} \quad (53)$$

where $J_{t,y}$ is the Jacobian of the transformation. This time treating both y and t as complex variables and applying partial fractioning, this last integral can be rewritten as:

$$\begin{aligned} & b_i = i(-2\pi i)^2 \int \frac{dt dy}{(2\pi)^4} J_{t,y} A_1^{(i)}(t, y) A_2^{(i)}(t, y) \\ &\rightarrow i(-2\pi i)^2 \int \frac{dt dy}{(2\pi)^4} J_{t,y} \left(\left[\text{Inf}_y \left[\text{Inf}_t A_1^{(i)} A_2^{(i)} \right] \right] (t, y) \right. \\ &\quad + \left[\text{Inf}_y \sum_{\{k\}} \left[\frac{\text{Res}_{t=t_k} A_1^{(i)} A_2^{(i)}}{t - t_k} \right] \right] (y) \\ &\quad + \sum_{\{j\}} \left[\frac{\text{Res}_{y=y_j} \left[\text{Inf}_t A_1^{(i)} A_2^{(i)} \right] (t)}{y - y_j} \right] \\ &\quad \left. + \sum_{\{j\}} \left[\frac{\text{Res}_{y=y_j} \sum_{\{k\}} \left[\frac{\text{Res}_{t=t_k} A_1^{(i)} A_2^{(i)}}{t - t_k} \right]}{y - y_j} \right] \right) \end{aligned} \quad (54)$$

The double $[\text{Inf}]$ terms are the pure bubble contributions to the double-cut, the single residue terms are triangle contributions and the double residue terms are box contributions. As before, the box contributions only give the scalar box coefficients and are therefore not of interest to extracting bubble coefficients. The triangle terms are not so easily dismissed. Unlike in the previous section, the parametrization of the loop momentum does not annihilate all tensor triangle contributions. Since tensor triangles can be decomposed into scalar triangles, bubbles and tadpoles, there is a contribution to the bubble coefficient from the single residue terms.

A. Pure Bubble Contributions to the Bubble Coefficient

To extract the pure bubble contribution to the coefficient, I must build an integral table for powers of both t and y . If I consider tensor bubbles of the form

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot a_4)^n}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)} \rightarrow (-2\pi i)^2 \gamma_{1\chi}^n \int \frac{dt dy}{(2\pi)^4} J_{t,y} t^n \quad (55)$$

For $n = 0$, this is just the scalar bubble and the result is of course equal to unity. For $n > 0$, the integrals vanish because, by Lorentz invariance, tensor bubbles have components made up of $K_1^{\mu_i}$ and the metric tensor $g^{\mu_i \mu_j}$, both of which annihilate products of $a_4^{\mu_1} \dots a_4^{\mu_n}$. Thus,

$$\begin{aligned} \int \frac{dt dy}{(2\pi)^4} J_{t,y} &= 1, \\ \int \frac{dt dy}{(2\pi)^4} J_{t,y} t^{n>0} &= 0. \end{aligned} \quad (56)$$

The integral table for values of y is somewhat more complicated. First, I form the auxiliary vector

$$\begin{aligned} \omega &= K_1^\perp - \frac{S_2}{\gamma_{1\chi}} \chi, \quad K_1 \cdot \omega = 0, \quad \omega^2 = -S_1, \\ \ell \cdot \omega &= \frac{1}{2} (S_1 + m_0^2 - m_1^2) - S_1 y \end{aligned} \quad (57)$$

Using ω , I find,

$$\begin{aligned} \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot \omega)^{2k-1}}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)} &= 0 \\ \Rightarrow \int dt dy J_{t,y} \left(\frac{S_1 + m_0^2 - m_1^2}{2} - S_1 y \right)^{2k-1} &= 0, \\ \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot \omega)^{2k}}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)} &= \frac{(2k)!}{2^k k!} (-S_1)^k B^{\{0^{2k}\}} \\ \Rightarrow \int dt dy J_{t,y} \left(\frac{S_1 + m_0^2 - m_1^2}{2S_1} - y \right)^{2k} &= \left(\frac{-1}{S_1} \right)^k \frac{(2k)!}{2^k k!} \tilde{B}^{\{0^{2k}\}}, \end{aligned} \quad (58)$$

where $B^{\{0^{2k}\}}$ is the Passarino-Veltman reduction term containing only products of the metric tensor (no factors of K_1^μ), and $\tilde{B}^{\{0^{2k}\}}$ is the corresponding coefficient of the scalar bubble in that term. One can avoid actually constructing the Passarino-

Veltman reduction by using another set of identities,

$$\begin{aligned} \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot a_3)^k (\ell \cdot a_4)^k}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)} &= (-1)^k 2^k k! \gamma_{1\chi}^k B^{\{0^{2k}\}} \\ \Rightarrow \int dt dy J_{t,y} \left(-\frac{m_0^2}{S_1} + y \frac{S_1 + m_0^2 - m_1^2}{S_1} - y^2 \right)^k &= \left(\frac{-1}{S_1} \right)^k 2^k k! \tilde{B}^{\{0^{2k}\}}, \end{aligned} \quad (59)$$

Eliminating the Passarino-Veltman coefficient, I find

$$\begin{aligned} \int \frac{dt dy}{(2\pi)^4} J_{t,y} &= 1, \\ \int \frac{dt dy}{(2\pi)^4} J_{t,y} \left(\frac{S_1 + m_0^2 - m_1^2}{2} - S_1 y \right)^{2k-1} &= 0, \\ \int \frac{dt dy}{(2\pi)^4} J_{t,y} \left[\frac{2^k k!}{(2k)!} \left(\frac{S_1 + m_0^2 - m_1^2}{2S_1} - y \right)^{2k} \right. \\ &\quad \left. - \frac{1}{2^k k!} \left(-\frac{m_0^2}{S_1} + y \frac{S_1 + m_0^2 - m_1^2}{S_1} - y^2 \right)^k \right] = 0. \end{aligned} \quad (60)$$

From these expressions, I can build an integral table for y :

$$\begin{aligned} B(n) &= \int \frac{dt dy}{(2\pi)^4} J_{t,y} y^n \\ &= \frac{(-1)^n}{n+1} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \left(\frac{-m_0^2}{S_1} \right)^i \left(\frac{S_1 + m_0^2 - m_1^2}{S_1} \right)^{n-2i}. \end{aligned} \quad (61)$$

B. Triangle Contributions to the Bubble Coefficient

It is also possible for there to be triangle contributions to the bubble coefficient. These will come from the single residue terms in Eq. (54). The residue terms correspond to an additional propagator going on-shell. Therefore, these terms can be obtained by applying an additional constraint to the double cut of the form,

$$\delta((\ell + K_2)^2 - m_2^2). \quad (62)$$

Without any loss of generality, y can be eliminated to satisfy the constraint, leaving t as the sole free parameter. The equa-

tion in y is

$$\begin{aligned}
0 = & y^2 \frac{S_1}{t \gamma_\chi} \langle \chi^- | K_2 | K_1^{\flat-} \rangle \\
& - 2y \left[K_1^{\flat-} \cdot K_2 - \frac{S_1}{\gamma_\chi} \chi \cdot K_2 \right. \\
& \left. + \frac{S_1 + m_0^2 - m_1^2}{2t \gamma_\chi} \langle \chi^- | K_2 | K_1^{\flat-} \rangle \right] \\
& - 2 \frac{S_1 + m_0^2 - m_1^2}{\gamma_\chi} \chi \cdot K_2 + S_2 + m_0^2 - m_2^2 \\
& + t \langle K_1^{\flat-} | K_2 | \chi^- \rangle - \frac{m_0^2}{t \gamma_\chi} \langle \chi^- | K_2 | K_1^{\flat-} \rangle
\end{aligned} \quad (63)$$

There are two solutions, $y = y_\pm$, which must be averaged over. Because the parametrization of the loop parameter is based upon the vectors K_1 and χ , instead of K_1 and K_2 , the tensor components of the triangle, that is, non-zero powers of t , will not identically vanish.

The integral table for t can be worked out by considering tensor integrals of the form

$$\begin{aligned}
& \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot a_4)^n}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)((\ell + K_2)^2 - m_2^2)} \\
& \rightarrow (-2\pi i)^3 \gamma_\chi^n \int \frac{dt dy}{(2\pi)^4} J'_t t^n.
\end{aligned} \quad (64)$$

Because a_4 is a null vector and is orthogonal to K_1 , if I perform Passarino-Veltman reduction, I find that only the $(K_2^\mu)^n$ component contributes to this integral. I then pick out the coefficient of the particular bubble I am looking at, $B_0(m_0, -K_1, m_1)$, from that Passarino-Veltman term. I find that

$$\begin{aligned}
T(n) &= \int \frac{dt dy}{(2\pi)^4} J'_t t^n \\
&= - \left(\frac{S_1}{2\gamma_\chi} \right)^n \frac{\langle \chi^- | K_2 | K_1^{\flat-} \rangle^n (K_1 \cdot K_2)^{n-1}}{\Delta^n(K_1, K_2)} \mathcal{C}_n,
\end{aligned} \quad (65)$$

where $\Delta(K_1, K_2)$ is the triangle Gram determinant and

$$\begin{aligned}
\mathcal{C}_0 &= 0, \\
\mathcal{C}_1 &= 1, \\
\mathcal{C}_2 &= \frac{3}{2} \left(\frac{S_1 + m_0^2 - m_1^2}{S_1} - \frac{S_2 + m_0^2 - m_2^2}{K_1 \cdot K_2} \right), \\
\mathcal{C}_3 &= \frac{5}{2} \left(\frac{S_1 + m_0^2 - m_1^2}{S_1} - \frac{S_2 + m_0^2 - m_2^2}{K_1 \cdot K_2} \right)^2 \\
&\quad - \frac{2 \Delta(K_1, K_2)}{3 (K_1 \cdot K_2)^2} \left[\left(\frac{S_1 + m_0^2 - m_1^2}{S_1} \right)^2 - 4 \frac{m_0^2}{S_1} \right].
\end{aligned} \quad (66)$$

C. The Bubble Coefficient

The complete bubble coefficient is found by combining the pure bubble and triangle contributions.

$$\begin{aligned}
b_i &= -i \left[\text{Inf}_y \left[\text{Inf}_t A_1^{(i)} A_2^{(i)} \right] \right] (t, y) \Big|_{t \rightarrow 0, y^n \rightarrow B(n)} \\
&\quad - \frac{1}{2} \sum_{\text{triangles } y=y_\pm} \sum \left[\text{Inf}_t \tilde{A}_1^{(i)} \tilde{A}_2^{(i)} \tilde{A}_3^{(i)} \right] (t) \Big|_{t^n \rightarrow T(n)},
\end{aligned} \quad (67)$$

where the $\tilde{A}_n^{(i)}$ are the amplitudes formed by cutting one more propagator in either $A_1^{(i)}$ or $A_2^{(i)}$.

VII. SINGLE PARTICLE CUTS AND THE EXTRACTION OF SCALAR TADPOLE COEFFICIENTS

The tadpole coefficients can be extracted by an extension of the same procedure. The only constraint that the loop momentum must satisfy is that $\ell^2 - m_0^2 = 0$. Satisfying this constraint leaves three free parameters in ℓ ,

$$\begin{aligned}
\ell^\mu &= y \chi^\mu + w \psi^\mu + \frac{t}{2} \langle \chi^- | \gamma^\mu | \psi^- \rangle + \alpha_4 \langle \psi^- | \gamma^\mu | \chi^- \rangle, \\
\alpha_4 &= \frac{1}{2t} \left(w y - \frac{m_0^2}{\gamma_\chi \psi} \right), \quad \gamma_\chi \psi = 2 \chi \cdot \psi,
\end{aligned} \quad (68)$$

where χ^μ and ψ^μ are arbitrary light-like momenta, since, in a tadpole configuration, there are no external momenta on which to base the parametrization.

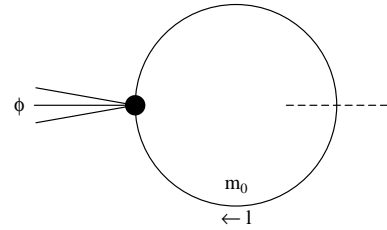


FIG. 5: a single cut.

The integral depicted in figure 5 is

$$\begin{aligned}
a_m A_0(m_0) &= i \int \frac{d^4 \ell}{(2\pi)^4} \frac{A^{(m)}(\emptyset; \ell)}{(\ell^2 - m_0^2)} \\
&\rightarrow i (-2\pi i) \int \frac{d^4 \ell}{(2\pi)^4} A^{(m)}(\emptyset; \ell) \delta(\ell^2 - m_0^2) \\
&= i (-2\pi i) \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} A^{(m)}(t, y, w),
\end{aligned} \quad (69)$$

where $A(\emptyset; \ell)$ indicates that there is no external momentum flowing in to the tadpole. I again treat the free parameters as

complex variables and apply partial fractioning to obtain,

$$\begin{aligned} & (-2\pi i) \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} A(t, y, w) = \\ & (-2\pi i) \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} \times \\ & ([\text{Inf}_w [\text{Inf}_y [\text{Inf}_t A]]] (t, y, w) + \dots). \end{aligned} \quad (70)$$

The triple [Inf] term will be the pure tadpole contribution to the tadpole, the double [Inf] - single [Res] terms will be the bubble contribution to the tadpole and the single [Inf] - double [Res] terms will be the triangle contribution to the tadpole. As before, the pure residue terms correspond to scalar box contributions and need not be considered.

A. Pure Tadpole Contributions to the Tadpole

To extract the pure tadpole contribution to the coefficient, I must build an integral table for powers of w , y , and t . This task is greatly simplified by the fact that tensor tadpole integrals vanish when the rank of the tensor is odd. Since there are no external momenta to project upon, the only non-vanishing tensors are proportional to products of the metric tensor. Further simplifications follow from the fact that the basis vectors are light-like and the only non-vanishing contractions are

$$\langle \chi^- | \gamma^\mu | \psi^- \rangle \langle \psi^- | \gamma_\mu | \chi^- \rangle = -4\chi \cdot \psi. \quad (71)$$

These facts lead to the conclusion that the only non-vanishing entries in the integral table are of the form

$$\int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} y^n w^n \neq 0, \quad (72)$$

where $n \geq 0$. All integrals involving non-zero powers of t vanish identically.

$$\begin{aligned} & \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot \chi)^k (\ell \cdot \psi)^k}{\ell^2 - m_0^2} = k! \left(\frac{\gamma_{\chi\psi}}{2} \right)^k A^{(2k)} \\ & \rightarrow (-2\pi i) \left(\frac{\gamma_{\chi\psi}}{2} \right)^{2k} \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} (wy)^k, \\ & \int \frac{d^4 \ell}{(2\pi)^4} \frac{\langle \chi^- | \ell | \psi^- \rangle^k \langle \psi^- | \ell | \chi^- \rangle^k}{\ell^2 - m_0^2} = k! (-2\gamma_{\chi\psi})^k A^{(2k)} \\ & \rightarrow (-2\pi i) (\gamma_{\chi\psi})^{2k} \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} \left(wy - \frac{m_0^2}{\gamma_{\chi\psi}} \right)^k. \end{aligned} \quad (73)$$

Together, these equations yield the result that

$$\begin{aligned} & (-2\pi i) \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} = 1, \\ & (-2\pi i) \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} \left[\left(wy - \frac{m_0^2}{\gamma_{\chi\psi}} \right)^k - (-wy)^k \right] = 0. \end{aligned} \quad (74)$$

A consistent solution to this set of equations is

$$D(k) = \int \frac{dt dy dw}{(2\pi)^4} J_{t,y,w} (wy)^k = \frac{1}{k+1} \left(\frac{m_0^2}{\gamma_{\chi\psi}} \right)^k \quad (75)$$

B. Bubble Contributions to the Tadpole

To obtain the bubble contributions to the tadpole, I add a second constraint,

$$\delta((\ell + K_1)^2 - m_1^2). \quad (76)$$

This constraint can be imposed by solving

$$\begin{aligned} 0 = & yw \frac{\langle \psi^- | K_1 | \chi^- \rangle}{t} + y2\chi \cdot K_1 + w2\psi \cdot K_1 \\ & + t \langle \chi^- | K_1 | \psi^- \rangle - \frac{m_0^2}{t\gamma_{\chi\psi}} \langle \psi^- | K_1 | \chi^- \rangle \\ & + S_1 + m_0^2 - m_1^2 \end{aligned} \quad (77)$$

for the values $w = w_0$ or $y = y_0$ and then constructing an integral table for t and y or t and w , respectively. In fact, one must average over the two solutions. The integral tables are constructed by considering tensor integrals of the form

$$\begin{aligned} & \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot \psi)^k \langle \psi^- | \ell | \chi^- \rangle^m}{(\ell^2 - m_0^2)((\ell + K_1)^2 - m_1^2)} \Big|_{w=w_0} \\ & = (K_1 \cdot \psi)^k \langle \psi^- | K_1 | \chi^- \rangle^m B^{\{1^{k+m}\}} \\ & \rightarrow (-2\pi i)^2 \left(\frac{\gamma_{\chi\psi}}{2} \right)^k (-\gamma_{\chi\psi})^m \int \frac{dt dy}{(2\pi)^4} J'_{t,y} y^k t^m, \\ & \int \frac{d^4 \ell}{(2\pi)^4} \frac{(\ell \cdot \chi)^k \langle \psi^- | \ell | \chi^- \rangle^m}{(\ell^2 - m_0^2)((\ell + K_1)^2 - m_1^2)} \Big|_{y=y_0} \\ & = (K_1 \cdot \chi)^k \langle \psi^- | K_1 | \chi^- \rangle^m B^{\{1^{k+m}\}} \\ & \rightarrow (-2\pi i)^2 \left(\frac{\gamma_{\chi\psi}}{2} \right)^k (-\gamma_{\chi\psi})^m \int \frac{dt dw}{(2\pi)^4} J'_{t,w} w^k t^m. \end{aligned} \quad (78)$$

The integral tables are:

$$\begin{aligned} E_w(k, n-k) &= \int \frac{dt dy}{(2\pi)^4} J'_{t,y} y^k t^{n-k} = \\ & - \frac{(-1)^n}{n+1} \frac{1}{S_1} \left(\frac{2\psi \cdot K_1}{\gamma_{\chi\psi}} \right)^k \left(-\frac{\langle \psi^- | K_1 | \chi^- \rangle}{\gamma_{\chi\psi}} \right)^{n-k} \\ & \times \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} \left(\frac{-m_0^2}{S_1} \right)^i \left(\frac{S_1 + m_0^2 - m_1^2}{S_1} \right)^{n-1-2i}, \\ E_y(k, n-k) &= \int \frac{dt dy}{(2\pi)^4} J'_{t,w} w^k t^{n-k} = \\ & - \frac{(-1)^n}{n+1} \frac{1}{S_1} \left(\frac{2\chi \cdot K_1}{\gamma_{\chi\psi}} \right)^k \left(-\frac{\langle \psi^- | K_1 | \chi^- \rangle}{\gamma_{\chi\psi}} \right)^{n-k} \\ & \times \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} \left(\frac{-m_0^2}{S_1} \right)^i \left(\frac{S_1 + m_0^2 - m_1^2}{S_1} \right)^{n-1-2i}. \end{aligned} \quad (79)$$

C. Triangle Contributions to the Tadpole

The triangle contributions to the tadpole come from adding a third constraint,

$$\delta((\ell + K_2)^2 - m_2^2). \quad (80)$$

The extra constraints can be imposed by simultaneously solving Eq. (77) and

$$\begin{aligned} 0 = & yw \frac{\langle \psi^- | K_2 | \chi^- \rangle}{t} + y2\chi \cdot K_2 + w2\psi \cdot K_2 \\ & + t \langle \chi^- | K_2 | \psi^- \rangle - \frac{m_0^2}{t\gamma_{\chi\psi}} \langle \psi^- | K_2 | \chi^- \rangle \\ & + S_2 + m_0^2 - m_2^2 \end{aligned} \quad (81)$$

for y and w . There are two solutions, $(w, y) = (w_1, y_1)$ and $(w, y) = (w_2, y_2)$, which must be averaged over. This leaves one free parameter t , for which I must derive an integral table. This can be done by considering tensor integrals of the form

$$\begin{aligned} & \int \frac{d^4\ell}{(2\pi)^4} \frac{\langle \psi^- | \ell | \chi^- \rangle^n}{(\ell^2 - m_0^2)((\ell + K_1)^2 - m_1^2)((\ell + K_2)^2 - m_2^2)} \\ & = \sum_{i=0}^n \langle \psi^- | K_1 | \chi^- \rangle^{n-i} \langle \psi^- | K_2 | \chi^- \rangle^i C^{\{1^{n-i}2^i\}} \\ & \rightarrow (-2\pi i)^3 (-\gamma_{\chi\psi})^n \int \frac{dt}{(2\pi)^4} J_t'' t^n, \end{aligned} \quad (82)$$

where $C^{\{1^{n-i}2^i\}}$ is the n -th rank Passarino-Veltman triangle coefficient whose Lorentz structure has $(n-i)$ powers of K_1 and i powers of K_2 . I only need the tadpole term from the Passarino-Veltman coefficients, so my integral table is:

$$\begin{aligned} F(n) &= \int \frac{dt}{(2\pi)^4} J_t'' t^n \\ &= \sum_{i=0}^n \frac{\langle \psi^- | K_1 | \chi^- \rangle^{n-i} \langle \psi^- | K_2 | \chi^- \rangle^i}{\gamma_{\chi\psi}^n} f_n(i) \end{aligned} \quad (83)$$

with the $f_n(i)$'s given by

$$\begin{aligned} f_0(0) &= 0, \\ f_1(0) &= f_1(1) = 0 \\ f_2(0) &= \frac{1}{4} \frac{K_1 \cdot K_2}{S_1 \Delta(K_1, K_2)}, \quad f_2(1) = -\frac{1}{2} \frac{1}{\Delta(K_1, K_2)}, \\ f_2(2) &= \frac{1}{4} \frac{K_1 \cdot K_2}{S_2 \Delta(K_1, K_2)}, \\ f_3(0) &= \left(\frac{1}{6} \frac{(S_1 + m_0^2 - m_1^2)(K_1 \cdot K_2)}{S_1^2 \Delta(K_1, K_2)} \right. \\ & \quad + \frac{1}{8} \frac{S_2 + m_0^2 - m_2^2}{S_1 \Delta(K_1, K_2)} + \frac{5}{24} \frac{S_2(S_2 + m_0^2 - m_2^2)}{\Delta^2(K_1, K_2)} \\ & \quad \left. - \frac{5}{24} \frac{S_2(S_1 + m_0^2 - m_1^2)(K_1 \cdot K_2)}{S_1 \Delta^2(K_1, K_2)} \right), \\ f_3(1) &= \left(\frac{1}{24} \frac{S_1 + m_0^2 - m_1^2}{S_1 \Delta(K_1, K_2)} \right. \\ & \quad - \frac{5}{24} \frac{(S_2 + m_0^2 - m_2^2)(K_1 \cdot K_2)}{\Delta^2(K_1, K_2)} \\ & \quad \left. + \frac{5}{24} \frac{S_2(S_1 + m_0^2 - m_1^2)}{\Delta^2(K_1, K_2)} \right), \\ f_3(2) &= f_3(1)|_{K_1 \leftrightarrow K_2, m_1 \leftrightarrow m_2}, \\ f_3(3) &= f_3(0)|_{K_1 \leftrightarrow K_2, m_1 \leftrightarrow m_2}. \end{aligned} \quad (84)$$

D. The Tadpole Coefficient

The complete tadpole coefficient is found by combining the pure tadpole, bubble and triangle contributions.

$$\begin{aligned} a_m &= \left[\text{Inf}_w \left[\text{Inf}_y \left[\text{Inf}_t A^{(m)} \right] \right] \right] (t, y, w) \Big|_{t \rightarrow 0, (w, y)^n \rightarrow D(n)} \\ & - i \sum_{\text{bubbles}} \left[\left[\text{Inf}_y \left[\text{Inf}_t \tilde{A}_1^{(m)} \tilde{A}_2^{(m)} \right] \right] \Big|_{w=w_0} \right] \Big|_{y^k t^m \rightarrow E_w(k, m)} \\ & + \left[\text{Inf}_w \left[\text{Inf}_t \tilde{A}_1^{(m)} \tilde{A}_2^{(m)} \right] \right] \Big|_{y=y_0} \Big|_{w^k t^m \rightarrow E_y(k, m)} \\ & - \frac{1}{2} \sum_{\text{triangles}} \sum_{(w, y) = (w_1, y_1)}^{(w_2, y_2)} \left[\text{Inf}_t \hat{A}_1^{(m)} \hat{A}_2^{(m)} \hat{A}_3^{(m)} \right] (t) \Big|_{t^n \rightarrow F(n)}, \end{aligned} \quad (85)$$

where the $\tilde{A}_n^{(m)}$ are the amplitudes formed by cutting a propagator in $A^{(m)}$ and $\hat{A}_n^{(m)}$ are the amplitudes formed by cutting two propagators in $A^{(m)}$.

VIII. COMMENTS

The expressions given above assume that all external momenta are massive and that all internal masses are distinct. They have been tested numerically by decomposing tensor integrals (through the fourth rank tensor box) using Passarino-Veltman techniques and those described here.

As long as the external momenta are massive, the only complication in allowing the internal masses to become degenerate comes from the tadpoles. In case of degeneracy, one must be careful not to over count the contributions from bubbles and triangles. This can clearly happen because the starting amplitude for the tadpole cut $A^{(m)}(0, \ell)$ is not tied to any external momenta. Thus, if one successively adds cuts to the tadpole configuration, one can reach the same bubble or triangle configuration from different starting points. It is safer to construct special cases for degenerate mass bubbles and triangles and simply sum over the distinct configurations.

For $S_1 \neq 0$ and $m_0 = m_1 \neq 0$,

$$\begin{aligned} E_w(k, n-k) &= \int \frac{dt dy}{(2\pi)^4} J_{t,y}' t^{n-k} = \\ & \frac{(-1)^n}{n+1} \frac{1}{S_1} \left(\frac{2\psi \cdot K_1}{\gamma_{\chi\psi}} \right)^k \left(-\frac{\langle \psi^- | K_1 | \chi^- \rangle}{\gamma_{\chi\psi}} \right)^{n-k} \\ & \times \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \left(\frac{-m_0^2}{S_1} \right)^i, \\ E_y(k, n-k) &= \int \frac{dt dy}{(2\pi)^4} J_{t,w}' w^k t^{n-k} = \\ & \frac{(-1)^n}{n+1} \frac{1}{S_1} \left(\frac{2\chi \cdot K_1}{\gamma_{\chi\psi}} \right)^k \left(-\frac{\langle \psi^- | K_1 | \chi^- \rangle}{\gamma_{\chi\psi}} \right)^{n-k} \\ & \times \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \left(\frac{-m_0^2}{S_1} \right)^i. \end{aligned} \quad (86)$$

Some formulæ must also be modified when external momenta are massless. In particular, the bubble integral is no longer an independent loop-integral function,

$$\begin{aligned} B_0(0, m_0, K_1, m_1)|_{S_1=0, m_0 \neq m_1} &= \frac{A_0(m_0) - A_0(m_1)}{m_0^2 - m_1^2}, \\ B_0(0, m_0, K_1, m_1)|_{S_1=0, m_0=m_1 \neq 0} &= (1 - \varepsilon) \frac{A_0(m_0)}{m_0^2}, \\ B_0(0, m_0, K_1, m_1)|_{S_1=0, m_0=m_1=0} &= 0. \end{aligned} \quad (87)$$

In addition to obviating the need to extract the coefficient of the massless bubbles, this identity alters the extraction of tadpole terms from bubbles and triangles. The expressions in Eq. (79) become for $S_1 = 0, m_0 \neq m_1$,

$$\begin{aligned} E_w(k, n-k) &= \frac{(-1)^k}{n+1} \left(\frac{m_0^2}{m_0^2 - m_1^2} \right)^{n+1} \\ &\quad \times \frac{(2\psi \cdot K_1)^k \langle \psi^- | K_1 | \chi^- \rangle^{n-k}}{m_0^2 \gamma_{\chi\psi}^n}, \\ E_y(k, n-k) &= \frac{(-1)^k}{n+1} \left(\frac{m_0^2}{m_0^2 - m_1^2} \right)^{n+1} \\ &\quad \times \frac{(2\chi \cdot K_1)^k \langle \psi^- | K_1 | \chi^- \rangle^{n-k}}{m_0^2 \gamma_{\chi\psi}^n}, \end{aligned} \quad (88)$$

and for $S_1 = 0, m_0 = m_1 \neq 0$,

$$\begin{aligned} E_w(k, n-k) &= \frac{(-1)^k}{n+1} \frac{(2\psi \cdot K_1)^k \langle \psi^- | K_1 | \chi^- \rangle^{n-k}}{m_0^2 \gamma_{\chi\psi}^n}, \\ E_y(k, n-k) &= \frac{(-1)^k}{n+1} \frac{(2\chi \cdot K_1)^k \langle \psi^- | K_1 | \chi^- \rangle^{n-k}}{m_0^2 \gamma_{\chi\psi}^n}, \end{aligned} \quad (89)$$

Clearly this last case is one in which one must be careful to count the contribution of this bubble only once, since it can be reached two different starting points.

Formulæ which depend upon Passarino-Veltman triangle coefficients also change significantly. Actually, Eq. (66) is valid in the $S_2 \rightarrow 0$ limit and does not appear in the $S_1 \rightarrow 0$ limit since, the bubble function $B_0(m_0, -K_1, m_1)|_{S_1=0}$ is not a basis integral. Eq. (84), however, changes dramatically as the external momenta vanish and the internal masses become degenerate. There are five different special cases to consider as one takes different combinations of $S_n = 0$ and $m_n = m_0$ (taking into account exchange symmetries). The full set of expressions is given in Appendix (A).

IX. CONCLUSIONS

I have described a method, based upon generalized unitarity, for computing the four-dimensional coefficients of scalar loop-integral functions in one-loop amplitudes in quantum field theory. The result is that the coefficients of the loop integrals are determined from the product of tree-level diagrams that appear at the vertices of the loop-integral itself.

The method is valid for arbitrary internal and external masses and can be applied to any one-loop calculation in quantum field theory. When combined with one-loop on-shell recursion relations, this procedure can be used to construct complete one-loop amplitudes.

One of the features of this method is that it can make use of the compact representation of tree-level amplitudes found by using helicity methods. This formalism, when combined with on-shell recursion relations to determine the rational terms, as envisioned in the unitarity bootstrap, shows great promise for constructing an automated system for generating one-loop amplitudes.

APPENDIX A: SPECIAL CASES FOR THE TRIANGLE CONTRIBUTION TO THE TADPOLE

1. Case 1: $S_1 = 0, m_1 \neq m_0$

Because the massless scalar bubble breaks into a sum of tadpoles, the special cases gives non-vanishing contributions to $F(1)$.

$$\begin{aligned}
 f_0(0) &= 0, \\
 f_1(0) &= \frac{1}{2} \frac{1}{(K_1 \cdot K_2)(m_0^2 - m_1^2)}, \quad f_1(1) = 0 \\
 f_2(0) &= -\frac{1}{4} \left(\frac{m_0^2}{(K_1 \cdot K_2)(m_0^2 - m_1^2)^2} + \frac{S_2 + m_0^2 - m_2^2}{(K_1 \cdot K_2)^2(m_0^2 - m_1^2)} \right. \\
 &\quad \left. - \frac{3}{2} \frac{S_2}{(K_1 \cdot K_2)^3} \right), \\
 f_2(1) &= -\frac{3}{8} \frac{1}{(K_1 \cdot K_2)^2}, \quad f_2(2) = \frac{1}{4} \frac{1}{S_2(K_1 \cdot K_2)}, \\
 f_3(0) &= \frac{1}{6} \frac{1}{(K_1 \cdot K_2)m_0^2} + \frac{1}{8} \frac{(S_2 + m_0^2 - m_2^2)}{m_0^2(K_1 \cdot K_2)^2} \\
 &\quad + \frac{1}{8} \frac{(S_2 + m_0^2 - m_2^2)^2}{m_0^2(K_1 \cdot K_2)^3} - \frac{25}{48} \frac{S_2(S_2 + m_0^2 - m_2^2)}{(K_1 \cdot K_2)^4} \\
 &\quad + \frac{5}{16} \frac{S_2^2(m_0^2 - m_1^2)}{(K_1 \cdot K_2)^5}, \\
 f_3(1) &= \frac{5}{16} \frac{S_2(m_1^2 - m_0^2)}{(K_1 \cdot K_2)^4} + \frac{1}{3} \frac{S_2 + m_0^2 - m_2^2}{(K_1 \cdot K_2)^3} \\
 &\quad - \frac{1}{8} \frac{m_0^2}{(m_0^2 - m_1^2)(K_1 \cdot K_2)^2}, \\
 f_3(2) &= -\frac{1}{4} \frac{m_1^2 - m_0^2}{(K_1 \cdot K_2)^3} - \frac{1}{24} \frac{S_2 + m_0^2 - m_2^2}{S_2(K_1 \cdot K_2)^2}, \\
 f_3(3) &= \frac{1}{8} \frac{(m_1^2 - m_0^2)}{S_2(K_1 \cdot K_2)^2} - \frac{1}{6} \frac{S_2 + m_0^2 - m_2^2}{S_2^2(K_1 \cdot K_2)}.
 \end{aligned} \tag{A1}$$

2. Case 2: $S_1 = 0, m_1 = m_0$

This is perhaps the most difficult case and can be tricky, since the invariant mass of the third external momentum, $K_3 = K_1 - K_2$ appears in the denominator. It can happen that $S_3 = 0$, in which case one should be looking at Case 4 (see Appendix (A 4)), and if one had arrived at this configuration by cutting the other leg of the triangle first, one would naturally have arrived there. Therefore, the formula below assumes that $S_3 = (K_1 - K_2)^2 \neq 0$.

$$\begin{aligned}
 f_0(0) &= 0, \\
 f_1(0) &= \frac{1}{2} \frac{1}{m_0^2(K_1 \cdot K_2)}, \\
 f_1(1) &= 0, \\
 f_2(0) &= -\frac{1}{4} \frac{1}{S_3(K_1 \cdot K_2)} - \frac{1}{4} \frac{(K_1 \cdot K_2) + 2m_0^2 + S_2 - m_2^2}{m_0^2(K_1 \cdot K_2)^2}, \\
 f_2(1) &= \frac{1}{4} \frac{1}{S_3(K_1 \cdot K_2)}, \\
 f_2(2) &= \frac{1}{4} \frac{1}{S_2(K_1 \cdot K_2)} - \frac{1}{4} \frac{1}{S_3(K_1 \cdot K_2)}, \\
 f_3(0) &= \frac{1}{2} \frac{7S_3 - 2(m_0^2 - m_2^2)}{S_3(K_1 \cdot K_2)^2} + \frac{1}{8} \frac{m_0^2 - m_2^2}{S_3(K_1 \cdot K_2)^2} \\
 &\quad + \frac{1}{24} \frac{4(K_1 \cdot K_2) + 14m_0^2 + 3S_2 - 3m_2^2}{m_0^2(K_1 \cdot K_2)^2} \\
 &\quad + \frac{1}{8} \frac{(S_2 - m_2^2)^2}{m_0^2(K_1 \cdot K_2)^3} + \frac{1}{24} \frac{19S_2 + 8m_0^2 - 11m_2^2}{(K_1 \cdot K_2)^3}, \\
 f_3(1) &= -\frac{1}{6} \frac{2S_3 - m_0^2 + m_2^2}{S_3^2(K_1 \cdot K_2)} - \frac{1}{12} \frac{5S_3 + m_0^2 - m_2^2}{S_3(K_1 \cdot K_2)^2}, \\
 f_3(2) &= \frac{1}{12} \frac{S_3 - 2(m_0^2 - m_2^2)}{S_3^2(K_1 \cdot K_2)} \\
 &\quad + \frac{1}{24} \frac{m_0^2 - m_2^2}{S_3(K_1 \cdot K_2)^2} - \frac{1}{24} \frac{m_0^2 - m_2^2}{S_2(K_1 \cdot K_2)^2}, \\
 f_3(3) &= \frac{1}{6} \frac{S_3 + m_0^2 - m_2^2}{S_3^2(K_1 \cdot K_2)} - \frac{1}{6} \frac{S_2 + m_0^2 - m_2^2}{S_2^2(K_1 \cdot K_2)}.
 \end{aligned} \tag{A2}$$

3. Case 3: $S_1 = S_2 = 0, m_1 \neq m_0, m_2 \neq m_0$

If both $S_1 = 0$ and $S_2 = 0$, the third external momentum $K_3 = K_1 - K_2$ must be massive, $S_3 = -2K_1 \cdot K_2 \neq 0$. In the limit that S_3 becomes small, the diagram describes a one-loop splitting amplitude, rather than a scattering amplitude.

$$\begin{aligned}
f_0(0) &= 0, \\
f_1(0) &= \frac{1}{2} \frac{1}{(m_0^2 - m_1^2)(K_1 \cdot K_2)}, \\
f_1(1) &= \frac{1}{2} \frac{1}{(m_0^2 - m_2^2)(K_1 \cdot K_2)}, \\
f_2(0) &= -\frac{1}{4} \frac{m_0^2}{(m_0^2 - m_1^2)^2(K_1 \cdot K_2)} - \frac{1}{4} \frac{m_0^2 - m_2^2}{(m_0^2 - m_1^2)(K_1 \cdot K_2)^2}, \\
f_2(1) &= -\frac{1}{2} \frac{1}{(K_1 \cdot K_2)^2}, \\
f_2(2) &= -\frac{1}{4} \frac{m_0^2}{(m_0^2 - m_2^2)^2(K_1 \cdot K_2)} - \frac{1}{4} \frac{m_0^2 - m_1^2}{(m_0^2 - m_2^2)(K_1 \cdot K_2)^2}, \\
f_3(0) &= \frac{1}{6} \frac{m_0^4}{(m_0^2 - m_1^2)^3(K_1 \cdot K_2)} + \frac{1}{8} \frac{m_0^2(m_0^2 - m_2^2)}{(m_0^2 - m_1^2)^2(K_1 \cdot K_2)^2} \\
&\quad + \frac{1}{8} \frac{(m_0^2 - m_2^2)^2}{(m_0^2 - m_1^2)(K_1 \cdot K_2)^3}, \\
f_3(1) &= -\frac{1}{8} \frac{m_0^2}{(m_0^2 - m_1^2)(K_1 \cdot K_2)^2} + \frac{3}{8} \frac{m_0^2 - m_2^2}{(K_1 \cdot K_2)^3}, \\
f_3(2) &= -\frac{1}{8} \frac{m_0^2}{(m_0^2 - m_2^2)(K_1 \cdot K_2)^2} + \frac{3}{8} \frac{m_0^2 - m_1^2}{(K_1 \cdot K_2)^3}, \\
f_3(3) &= \frac{1}{6} \frac{m_0^4}{(m_0^2 - m_2^2)^3(K_1 \cdot K_2)} + \frac{1}{8} \frac{m_0^2(m_0^2 - m_1^2)}{(m_0^2 - m_2^2)^2(K_1 \cdot K_2)^2} \\
&\quad + \frac{1}{8} \frac{(m_0^2 - m_1^2)^2}{(m_0^2 - m_2^2)(K_1 \cdot K_2)^3}.
\end{aligned} \tag{A3}$$

4. Case 4: $S_1 = S_2 = 0, m_1 = m_0, m_2 \neq m_0$

Again, in a scattering amplitude, $S_3 = -2K_1 \cdot K_2 \neq 0$

$$\begin{aligned}
f_0(0) &= 0, \\
f_1(0) &= \frac{1}{2} \frac{1}{m_0^2(K_1 \cdot K_2)}, \\
f_1(1) &= \frac{1}{2} \frac{1}{(m_0^2 - m_2^2)(K_1 \cdot K_2)}, \\
f_2(0) &= -\frac{1}{4} \frac{(K_1 \cdot K_2) - m_2^2}{m_0^2(K_1 \cdot K_2)^2} - \frac{3}{8} \frac{1}{(K_1 \cdot K_2)^2}, \\
f_2(1) &= -\frac{1}{4} \frac{1}{(K_1 \cdot K_2)^2}, \\
f_2(2) &= -\frac{1}{4} \frac{m_0^2}{(m_0^2 - m_2^2)^2(K_1 \cdot K_2)} + \frac{1}{8} \frac{1}{(K_1 \cdot K_2)^2}, \\
f_3(0) &= \frac{1}{6} \frac{1}{m_0^2(K_1 \cdot K_2)} + \frac{7}{24} \frac{1}{(K_1 \cdot K_2)^2} \\
&\quad - \frac{1}{8} \frac{m_2^2((K_1 \cdot K_2) + m_0^2 - m_2^2)}{m_0^2(K_1 \cdot K_2)^3} + \frac{11}{48} \frac{m_0^2 - m_2^2}{(K_1 \cdot K_2)^3}, \\
f_3(1) &= \frac{1}{8} \frac{2(K_1 \cdot K_2) - m_0^2 + m_2^2}{(K_1 \cdot K_2)^3}, \\
f_3(2) &= -\frac{1}{8} \frac{m_0^2}{(m_0^2 - m_2^2)(K_1 \cdot K_2)^2} - \frac{1}{16} \frac{m_0^2 - m_2^2}{(K_1 \cdot K_2)^3}, \\
f_3(3) &= \frac{1}{6} \frac{m_0^4}{(m_0^2 - m_2^2)^3} - \frac{1}{24} \frac{2(K_1 \cdot K_2) - m_0^2 + m_2^2}{(K_1 \cdot K_2)^3},
\end{aligned} \tag{A4}$$

5. Case 5: $S_1 = S_2 = 0, m_2 = m_1 = m_0$

If both $S_1 = 0$ and $S_2 = 0$, and all the internal masses are degenerate,

$$\begin{aligned}
f_0(0) &= 0, \\
f_1(0) &= f_1(1) = \frac{1}{2} \frac{1}{m_0^2(K_1 \cdot K_2)}, \\
f_2(0) &= f_2(2) = -\frac{1}{4} \frac{1}{m_0^2(K_1 \cdot K_2)}, \\
f_2(1) &= 0, \\
f_3(0) &= f_3(3) = \frac{1}{6} \frac{1}{m_0^2(K_1 \cdot K_2)} + \frac{1}{12} \frac{1}{(K_1 \cdot K_2)^2}, \\
f_3(1) &= f_3(2) = -\frac{1}{4} \frac{1}{(K_1 \cdot K_2)^2}.
\end{aligned} \tag{A5}$$

6. Case 6: $S_1 = 0, S_2, S_3 \neq 0, m_2 = m_1 = m_0$

If only $S_1 = 0$, but all the internal masses are degenerate,

$$\begin{aligned}
 f_0(0) &= 0, \\
 f_1(0) &= \frac{1}{2} \frac{1}{m_0^2 (K_1 \cdot K_2)}, \\
 f_1(1) &= 0, \\
 f_2(0) &= -\frac{1}{4} \frac{1}{m_0^2 (K_1 \cdot K_2)} - \frac{1}{4} \frac{S_2}{m_0^2 (K_1 \cdot K_2)^2}, \\
 f_2(1) &= f_2(2) = 0, \\
 f_3(0) &= \frac{1}{6} \frac{1}{m_0^2 (K_1 \cdot K_2)} + \frac{1}{6} \frac{1}{S_3 (K_1 \cdot K_2)} \\
 &\quad + \frac{1}{6} \frac{1}{(K_1 \cdot K_2)^2} + \frac{1}{8} \frac{S_2}{m_0^2 (K_1 \cdot K_2)^2} \\
 &\quad + \frac{1}{3} \frac{S_2}{(K_1 \cdot K_2)^3} + \frac{1}{8} \frac{S_2^2}{m_0^2 (K_1 \cdot K_2)^3}, \\
 f_3(1) &= -\frac{1}{6} \frac{1}{S_3 (K_1 \cdot K_2)} - \frac{1}{3} \frac{1}{(K_1 \cdot K_2)^2}, \\
 f_3(2) &= \frac{1}{6} \frac{1}{S_3 (K_1 \cdot K_2)}, \\
 f_3(3) &= \frac{1}{6} \frac{1}{S_2 (K_1 \cdot K_2)} - \frac{1}{6} \frac{1}{S_3 (K_1 \cdot K_2)}.
 \end{aligned} \tag{A6}$$

7. Case 7: $S_1, S_2, S_3 \neq 0, m_2 = m_1 = m_0$

If all external masses are non-zero but all the internal masses are degenerate,

$$\begin{aligned}
 f_0(0) &= 0, \\
 f_1(0) &= f_1(1) = 0, \\
 f_2(0) &= f_2(1) = f_2(2) = 0, \\
 f_3(0) &= \frac{1}{6} \frac{S_2 - (K_1 \cdot K_2)}{S_3 \Delta(K_1, K_2)} \\
 &\quad + \frac{1}{6} \frac{S_2}{\Delta(K_1, K_2) (K_1 \cdot K_2)} + \frac{1}{6} \frac{1}{S_1 (K_1 \cdot K_2)}, \\
 f_3(1) &= \frac{1}{6} \frac{(K_1 \cdot K_2) - S_2}{S_3 \Delta(K_1, K_2)}, \\
 f_3(2) &= \frac{1}{6} \frac{S_2 - S_3 - (K_1 \cdot K_2)}{S_3 \Delta(K_1, K_2)}, \\
 f_3(3) &= \frac{1}{6} \frac{S_2 S_3 - S_2^2 + (S_2 + S_3)(K_1 \cdot K_2)}{S_2 S_3 \Delta(K_1, K_2)}.
 \end{aligned} \tag{A7}$$

Acknowledgments: I would like to thank Zvi Bern for helpful conversations in the early part of this work. This research was supported by the U. S. Department of Energy under Contract No. DE-AC02-98CH10886.

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